

## Branching of the Falkner-Skan solutions for $\lambda < 0$

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(Received May 17, 1982)

### SUMMARY

The Falkner-Skan equation  $f''' + ff'' + \lambda(1 - f'^2) = 0$ ,  $f(0) = f'(0) = 0$ , is discussed for  $\lambda < 0$ . Two types of problems, one with  $f'(\infty) = 1$  and another with  $f'(\infty) = -1$ , are considered. For  $\lambda = 0^-$  a close relation between these two types is found. For  $\lambda < -1$  both types of problem allow multiple solutions which may be distinguished by an integer  $N$  denoting the number of zeros of  $f' - 1$ . The numerical results indicate that the solution branches with  $f'(\infty) = 1$  and those with  $f'(\infty) = -1$  tend towards a common limit curve as  $N$  increases indefinitely. Finally a periodic solution, existing for  $\lambda < -1$ , is presented.

### 1. Introduction

An important class of similarity solutions in boundary-layer theory is governed by the Falkner-Skan equation

$$f''' + ff'' + \lambda(1 - f'^2) = 0, \quad (1)$$

with the usual boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \quad (2)$$

This equation has been introduced about half a century ago [1]. An account of its physical significance is given by Schlichting [2]. Existence and uniqueness of the solutions of (1) + (2) have been discussed in many papers. Therefore let us first give a summary of the results obtained previously.

It has been shown that a unique solution exists for  $\lambda > 0$  under the additional requirement

$$0 < f' < 1 \quad (\eta > 0), \quad (3)$$

see for instance the monograph by Hartman [3]. In case of  $0 \leq \lambda \leq 1$  the restriction (3) can be removed, as proved by Coppel [4] and Craven and Peletier [5]. For  $\lambda > 1$  numerical evidence, supplied by Craven and Peletier [6], suggests that solutions of (1) + (2) exist which do not satisfy (3).

If  $\lambda < 0$  the situation is more complicated. It is known that there exists a number  $\lambda^* = -0.1988 \dots$  with the following properties.

- (i) For  $\lambda^* \leq \lambda < 0$ , a unique solution of (1) + (2) + (3) with  $f' \rightarrow 1$  exponentially exists, but algebraically decaying solutions exist also; see Iglisch and Kemnitz [7] and Hartman [8].
- (ii) For  $\lambda^* < \lambda < 0$  an additional, unique, exponentially decaying solution of (1) + (2) with  $f''(0) < 0$  exists. The existence and uniqueness of this reversed flow solution, first discussed by Stewartson [9], is proved by Hastings [10].
- (iii) For  $\lambda < \lambda^*$  no solutions of (1) + (2) exist which satisfy (3). But Libby and Liu [11] have presented exponentially decaying solutions which exhibit overshoot, i.e.  $f' > 1$  for some  $\eta$ . An existence proof of the Libby and Liu branches has recently been given by Troy [12].

A graphical presentation of these previously known branches for  $\lambda < 0$  is given in Figure 1.

The Falkner-Skan equation has also been studied subject to a second set of boundary conditions, namely

$$f(0) = f'(0) = 0, \quad f'(\infty) = -1. \tag{4}$$

Already in 1954 Stewartson [9] mentioned solutions satisfying (4). From that time applications have been presented by various authors [13–17]. Goldstein [13] has given the first, heuristic, discussion of existence and uniqueness. A rigorous treatment can be found in Ten Raa, et al. [16], where for  $\lambda < 0$  uniqueness is proved under the additional restriction  $f'' > 0$ . The latter restriction can be weakened as shown by Veldman and van de Vooren [18]. They proved existence and uniqueness of a solution of (1) + (4) with  $\lambda < 0$  under a restriction similar to (3), viz.

$$-1 < f' < 0 \quad (\eta > 0). \tag{5}$$

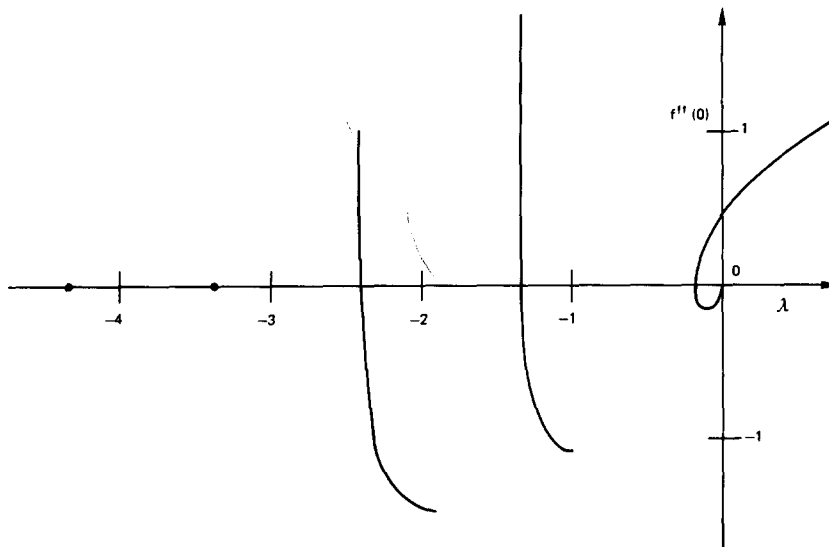


Figure 1.  $(\lambda, f''(0))$ -plane of solutions of the Falkner-Skan equation obtained by previous investigators.

Moreover the latter authors proved that for any solution of (1) + (4) the boundary condition at infinity is approached algebraically, i.e. as  $\eta \rightarrow \infty$

$$f'(\eta) \sim -1 + c\eta^{2\lambda} (c > 0).$$

Finally, Coppel [4] has proved that for  $\lambda \geq 0$  no solutions of (1) + (4) exist.

It is the objective of this paper to present for  $\lambda < 0$  new solutions of (1) + (2) or (1) + (4) displaying a close relation between these two problems. Curves in the  $(\lambda, f''(0))$  -plane representing either solutions of (1) + (2) or solutions of (1) + (4) will be studied. It is found that at  $\lambda = 0$  the curve corresponding to the solutions of (1) + (4) + (5) is tangent to the curve corresponding with the Stewartson solutions of (1) + (2).

We will extend the Libby and Liu branches towards higher values of  $\lambda$ . Moreover the paper will show multiple solutions of (1) + (4) with  $\lambda < -1$  which do not satisfy (5). There is a definite possibility that the corresponding branches in the  $(\lambda, f''(0))$  -plane, as well as the Libby and Liu branches, all start from a giant branching point  $B$  at  $\lambda = -1, f''(0) = -1.0863 \dots$ . The latter value satisfies a transcendental equation containing a parabolic cylinder function. Finally for  $\lambda < -1$  periodic solutions of (1) will be presented.

## 2. Reversed flow solutions for $\lambda = 0^-$

The investigation reported here is an outgrowth of a study on interacting boundary layers exhibiting regions with reversed flow. To describe these regions by means of an integral method solutions corresponding with the Stewartson branch are required for values of  $\lambda$  very close to zero [19].

An asymptotic theory describing these solutions has been presented by Brown and Stewartson [20]. At the time no numerical solutions with small enough values of  $-\lambda$  to check the asymptotic theory were available. We will present the comparison between the asymptotic theory and the numerical results however.

In short the asymptotic behaviour is as follows. Near the wall exists a region of size  $O((- \lambda)^{-1/4})$  where

$$f(\eta) \sim (-\lambda)^{1/4} F(Y), \quad Y = (-\lambda)^{1/4} \eta. \quad (6a)$$

$F(Y)$  is the unique [21] solution of

$$F''' + FF'' = 1, \quad F(0) = F'(0) = 0, \quad (6b)$$

which is negative for all values of  $Y > 0$ . Further  $f$  possesses one zero at  $\eta = \eta^*$ , a large distance from the wall. Near  $\eta = \eta^*$  we have

$$f(\eta) \sim g(\zeta), \quad \zeta = \eta - \eta^*, \quad (7a)$$

where  $g(\xi)$  represents a shear layer. This Chapman function  $g(\xi)$  satisfies

$$g''' + gg'' = 0, \quad g'(-\infty) = 0, \quad g(0) = 0, \quad g'(\infty) = 1. \tag{7b}$$

Matching (6a) and (7a) Brown and Stewartson [20] showed that  $\eta^*$  approximately satisfies

$$\eta^*(-\lambda)^{1/2} [2 \log ((-\lambda)^{1/4} \eta^*)]^{1/2} = -g(-\infty) = 0.87575. \tag{8}$$

Further, solving (6b) one finds

$$f''(0) \sim (-\lambda)^{3/4} F''(0) = -1.54400 (-\lambda)^{3/4}. \tag{9}$$

Table 1 gives a comparison between (8) and (9) and the numerical results of Oskam [19]. Taking into account that (8) predicts  $\eta^*$  with an error of at least  $O((-\lambda)^{-1/4})$  and that (9) gives  $f''(0)$  with an error  $O((-\lambda)^{7/4})$  [20] a perfect agreement is found between the asymptotic theory and the numerical results.

### 3. Solutions with $f' \rightarrow -1$

Calculating solutions corresponding to the Stewartson branch we observed nearby lying solutions of (1) + (4). Table 2 gives the value of  $f''(0)$  of the unique solution of (1) + (4) + (5) for a range of  $\lambda$ -values. Especially interesting is the behaviour of  $f''(0)$  as  $\lambda \rightarrow 0^-$ . It is very much like the behaviour of the Stewartson branch, as is apparent from the comparison with the asymptotic result (9) in Table 2.

Indeed, as asymptotic analysis set up along the lines of Brown and Stewartson [20], i.e.  $\lambda \rightarrow 0^-$ , leads to a wall region where the solution is governed by (6) again. Thus as  $\lambda \rightarrow 0^-$  the Stewartson branch of solutions of (1) + (2) and the branch of solutions of (1) + (4) + (5) are closely related: they have the same asymptotic behaviour near the wall given by (6).

In the other limit, as  $\lambda \rightarrow -\infty$ , the solution of (1) + (4) + (5) can be related to an analytical solution. Let

Table 1. A comparison of numerically obtained Falkner-Skan results with asymptotic theory for  $\lambda \rightarrow 0^-$ .

$\lim_{\eta \rightarrow \infty} \{\eta - f(\eta)\}$	$\lambda$	$f''(0)$		$\eta^*$	
		present	Eq. (9)	present	Eq. (8)
400	-1.01638 E-6	-4.94244 E-5	-4.94244 E-5	399.6	387.7
150	-9.50592 E-6	-2.64327 E-4	-2.64328 E-4	149.6	140.2
50	-1.31696 E-4	-1.89795 E-3	-1.89813 E-3	49.6	43.5
22	-1.19287 E-3	-9.79506 E-3	-9.91042 E-3	21.6	16.8
10	-1.52636 E-2	-5.52648 E-2	-6.70486 E-2	9.6	5.9

Table 2. Behaviour of  $f''(0)$  for Falkner-Skan solutions ( $N = 0$ ) with  $f'(\infty) = -1$ , compared with asymptotic theories for  $\lambda \rightarrow 0^-$  and for  $\lambda \rightarrow -\infty$ .

$\lambda$	$f''(0)$		
	present	Eq. (9)	Eq. (12)
-0.0001	-0.0015439	-0.0015440	
-0.001	-0.008676	-0.008683	
-0.01	-0.04848	-0.04883	
-0.1	-0.25752	-0.27457	
-0.2	-0.41113		
-0.6	-0.81202		
-1.0	-1.08638		
-1.2	-1.20150		
-1.6	-1.40457		
-2.0	-1.58230		-1.63299
-5.0	-2.54911		-2.58199
-10.0	-3.62805		-3.65148
-50.0	-8.15443		-8.16497
-100.00	-11.53955		-11.54701

$$\lambda = 2m/(m + 1), \quad \eta = [(m + 1)/2]^{1/2}s \quad \text{and} \quad f(\eta) = [(m + 1)/2]^{1/2}\hat{f}(s) \quad (10)$$

then  $\hat{f}(s)$  satisfies

$$\hat{f}''' + \frac{m + 1}{2}\hat{f}\hat{f}'' + m(1 - \hat{f}'^2) = 0, \quad \hat{f}(0) = \hat{f}'(0) = 0,$$

$$\hat{f}'(\infty) = 1. \quad (11)$$

Taking  $m = -1^+$ , corresponding to  $\lambda \rightarrow -\infty$ , (11) has the monotonic solution [2]

$$\hat{f}'(s) = -3 \tanh^2(2^{-1/2}s + s_0) + 2, \quad s_0 = \tanh^{-1}(\frac{2}{3})^{1/2},$$

with  $\hat{f}''(0) = -(\frac{4}{3})^{1/2}$ . Thus, assuming  $\hat{f}''(0)$  to be a continuous function of  $m$  for  $m = -1^+$ , we have

$$\lim_{\lambda \rightarrow -\infty} (-\lambda)^{-1/2}f''(0) = \lim_{m \rightarrow -1^+} (-m)^{-1/2}\hat{f}''(0) = -(\frac{4}{3})^{1/2}.$$

This implies that for large values of  $-\lambda$  the solution of (1) + (4) + (5) satisfies

$$f''(0) \sim -(-4\lambda/3)^{1/2}. \quad (12)$$

The results in Table 2 confirm this behaviour.

#### 4. An analytical solution for $\lambda = -1$

If  $\lambda = -1$  a solution of (1) + (4) can be calculated analytically. Two integrations of (1) yield the Riccati equation

$$f' + \frac{1}{2}f^2 = \eta f''(0) + \frac{1}{2}\eta^2.$$

Introduction of the new variables  $\xi = \eta + f''(0)$  and  $f(\eta) = 2w'(\xi)/w(\xi)$  leads to the Weber equation

$$w'' - (\frac{1}{4}\xi^2 + a)w = 0,$$

where  $a = -\frac{1}{4}f''(0)^2$ . The general solution can be written in terms of parabolic cylinder functions (Miller's notation will be used [22])

$$w(\xi) = c_1 U(a, \xi) + c_2 V(a, \xi).$$

As  $\xi \rightarrow \infty$ ,  $U(a, \xi)$  is decreasing exponentially, but  $V(a, \xi)$  is increasing exponentially. Thus, if  $c_2 \neq 0$ ,  $w(\xi)$  is an exponentially increasing function. As discussed by Yang and Chien [23] and Moulden [24], this corresponds to  $f' \rightarrow 1$  as  $\eta \rightarrow \infty^-$ . However, in both studies the case  $c_2 = 0$  has been overlooked. As is easily verified this case corresponds to  $f' \rightarrow -1$ . Moreover, as we will see in the next section, it will play a central role in the solutions of (1) + (2).

Having chosen  $c_2 = 0$ , the still unknown value of  $f''(0)$  can be found by imposing the boundary conditions on  $f$  at  $\eta = 0$ . This gives the transcendental equation

$$U'(a, f''(0)) = 0, \quad a = -\frac{1}{4}f''(0)^2. \quad (13)$$

Using the formulas and tables given in [22] it can be verified that the value  $f''(0) = -1.08638$ , obtained in Table 2, satisfies this equation. An analytical proof that this is the only solution of (13) has not been found yet.

#### 5. Multiple solutions for $\lambda < -1$

In 1966 Libby and Liu [11] presented exponentially decaying solutions of (1) + (2) for  $\lambda < -1$ . These solutions do not satisfy (3) as they possess regions where  $f' > 1$ . Recently, Troy [12] has shown that there is a sequence of branches of solutions such that  $f' - 1$  has precisely  $N$  zeros for each natural number  $N$ ; however, Troy did not indicate where one may find these solutions in the  $(\lambda, f''(0))$ -plane.

Libby and Liu [11] already conjectured that their first branch (one zero of  $f' - 1$ ,  $N = 1$ ) begins at  $\lambda = -1$ ,  $f''(0) \approx -1.09$ . This branch ends with a vertical asymptote at  $\lambda = -2$ , as  $f''(0) \rightarrow \infty$ ; see also Steinheuer [25]. We have extended this branch of solutions towards  $\lambda = -1$ . Some corresponding velocity profiles  $f'$  are presented in Figure 2. Numerical values of

$f''(0)$  may be found in Table 3. An extrapolation of the results from the three largest values of  $\lambda$  gives an estimate of  $-1.0864$  for  $f''(0)$  at  $\lambda = -1$ . Comparing this value with the analytical solution for  $\lambda = -1$  from the previous section suggests that at the point  $B = (-1, -1.08638)$  the first Libby and Liu branch coincides with the branch of solutions with  $f' \rightarrow -1$ . However, due to the limited word length of computers, we cannot isolate the Libby and Liu solution for  $\lambda$  arbitrarily close to  $-1$ , because the problem is ill-posed for  $\lambda \rightarrow -1$ .

The ill-posedness manifests itself even stronger if one intends to find the origin of the second and subsequent branches of Libby and Liu. They have pursued the second branch up to  $\lambda \approx -1.95$ . In Table 3 we give some results for  $\lambda$  up to  $-1.2$ . Note that at  $\lambda = -1.2$ ,  $f''(0)$  differs less than  $5 \cdot 10^{-5}$  from the value of  $f''(0)$  of the solution with  $f' \rightarrow -1$  (Table 2).

For  $\lambda = -2$  we have made a systematic search for other Libby and Liu branches. Several of them were found. The first five have been isolated; the corresponding values of  $f''(0)$  are given in Table 4. Note that  $f''(0)$  is a decreasing function of the number of zeros of  $f' - 1$ . Velocity profiles are shown in Figure 3.

During this search at  $\lambda = -2$  we also encountered solutions of (1) + (4) which do not satisfy (5). Like the Libby and Liu solutions they exhibit overshoot. Some velocity profiles are given in Figure 4; the corresponding values of  $f''(0)$  may be found in Table 4. It is noted

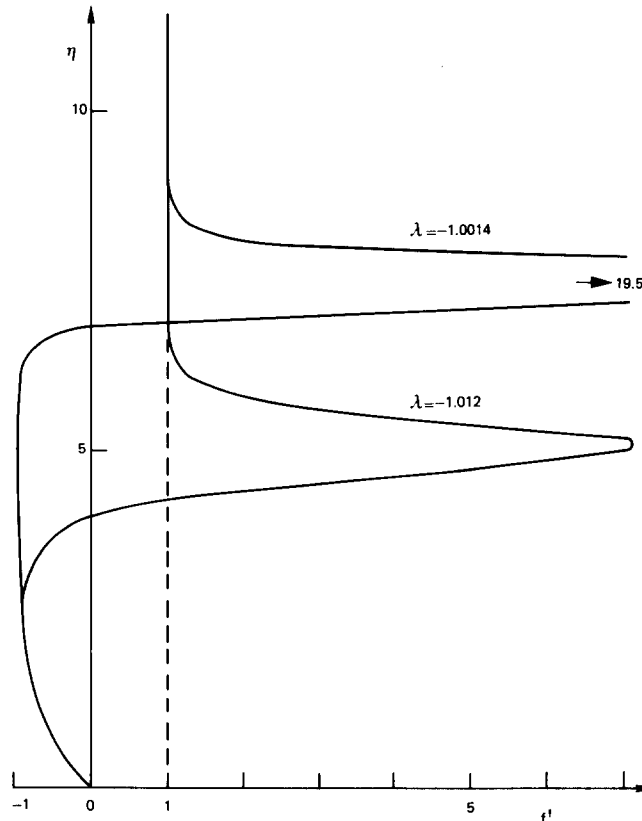


Figure 2. Falkner-Skan solutions on the first Libby and Liu branch near  $\lambda = -1$  ( $N = 1, f''(\infty) = 1$ ).

Table 3. Coordinates of the first two Libby and Liu branches.

First branch ( $N = 1$ )		Second branch ( $N = 2$ )	
$\lambda$	$f''(0)$	$\lambda$	$f''(0)$
-1.34742	0	-2	-1.46100
-1.26855	-0.72317	-1.92157	-1.47118
-1.09221	-1.05508	-1.84615	-1.46611
-1.02572	-1.09477	-1.70371	-1.43134
-1.02115	-1.09482	-1.47826	-1.34149
-1.00707	-1.09055	-1.33333	-1.27177
-1.00300	-1.08819	-1.25203	-1.22955
-1.00143	-1.08724	-1.20000	-1.20146

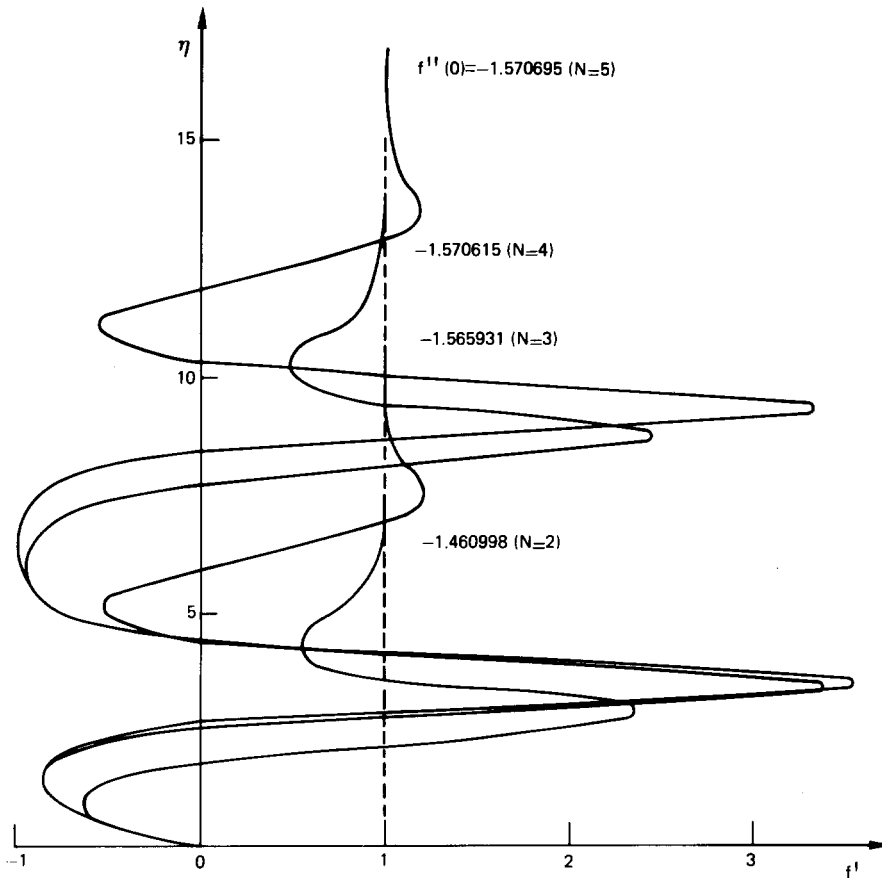


Figure 3. Falkner-Skan solutions on the second through fifth Libby and Liu branch for  $\lambda = -2$  ( $f'(\infty) = 1$ ).



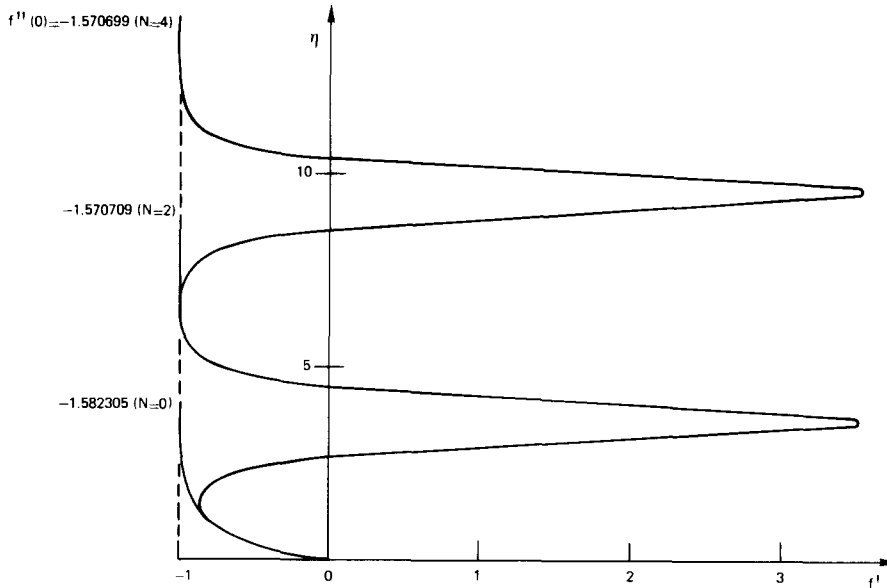


Figure 4. Falkner-Skan solutions for  $\lambda = -2$  ( $f'(\infty) = -1$ ),  $N = 0, 2, 4$ .

Table 4. Values of  $f''(0)$  of some Falkner-Skan solutions for  $\lambda = -2$ .

$f'(\infty)$	number of zeros of $f' - 1$					
	$N = 0$	1	2	3	4	5
+ 1	—	$\infty$	-1.46100	-1.56593	-1.57061	-1.57070
-1	-1.58230	—	-1.57071	—	-1.57070	—

that  $f''(0)$  of the solutions with  $f'(\infty) = -1$  is an increasing function of the number of zeros of  $f' - 1$ . Further the regular structure of the newly found oscillating solutions is worth mentioning. The ‘humps’ in the velocity profile all have about the same size and shape. We consider it likely that this can be related to the existence of a periodic solution of (1); the latter will be discussed in the next section.

At other values of  $\lambda < -1$  the same behaviour has been found. However, for  $-1 \leq \lambda < 0$  no sign could be found of solutions other than the ones already mentioned. Thus, it seems that for  $\lambda < -1$  a complicated branching process occurs, see Figure 5. A more complete picture of the branching process is given in Figure 6, where the scaling from (10) has been used. The solution branches with  $f'(\infty) = -1$  for  $N = 2, 4, \dots$  are lying very close together (see Figure 5). For increasing  $N$  a limit curve is approached, which on the scale of Figure 6 cannot be distinguished from the branch with  $N = 2$ . Figure 6 further shows the first seven Libby and Liu branches. For increasing  $\lambda (< -1)$  these branches, with  $f'(\infty) = 1$ , are seen to converge to the limit curve mentioned above on which  $f'(\infty) = -1$ .

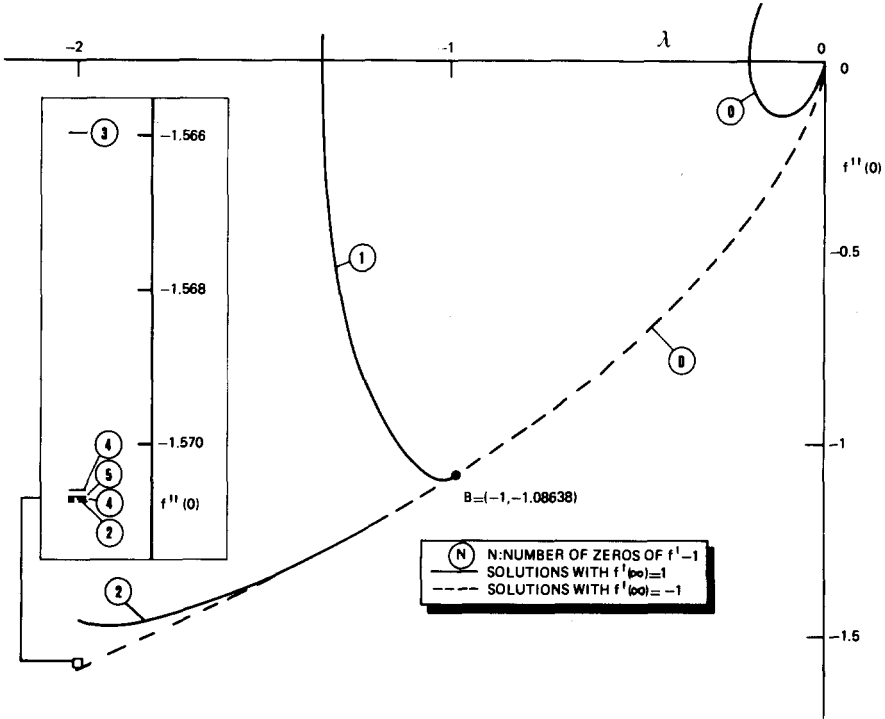


Figure 5. Falkner-Skan branches for  $\lambda < 0$ .

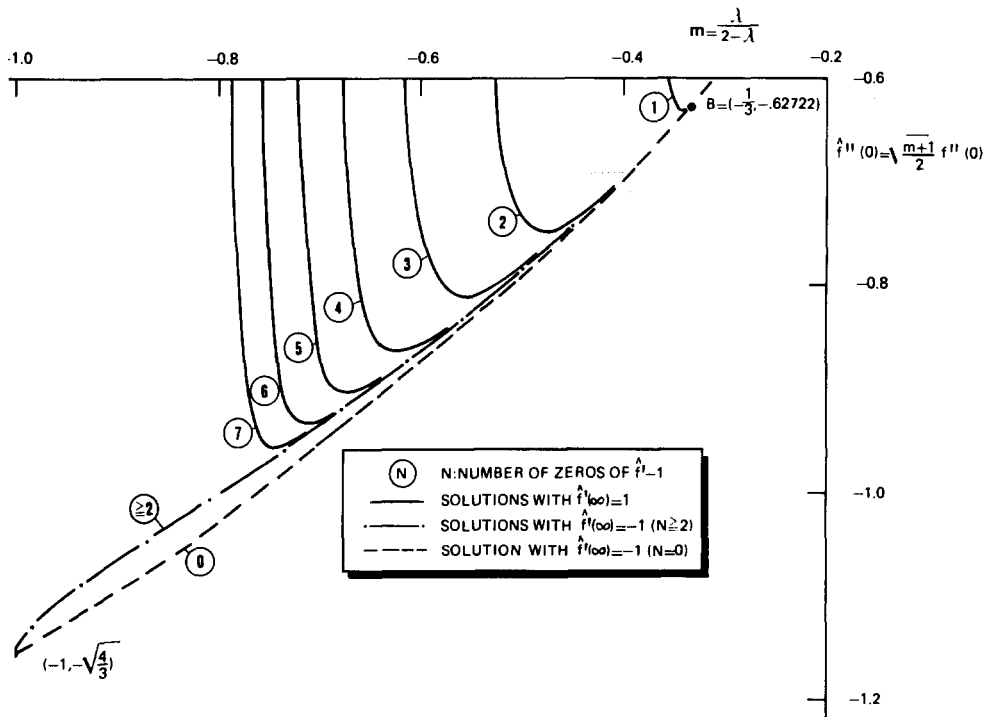


Figure 6. Falkner-Skan branches for  $m > -1$ .

6. Periodic solutions for  $\lambda < -1$

The regular structure of the newly found oscillating solutions suggests the existence of a periodic solution of the Falkner-Skan equation. Indeed, for  $\lambda < -1$  we have found numerical evidence of such a periodic solution, with period  $\eta_p > 0$ , which satisfies

$$f(0) = f''(0) = f(\eta_p) = f''(\eta_p) = 0; \tag{14}$$

moreover,  $f$  is antisymmetric with respect to  $\eta = \frac{1}{2}\eta_p$ . Figure 7 shows some periodic solutions. A few quantities corresponding with these numerical solutions can be found in Table 5, together with some asymptotic results for  $\lambda \rightarrow -1^-$ , to be described next. For  $\lambda \rightarrow -1^-$  an asymptotic description in terms of  $\epsilon = -1 - \lambda$  can be formulated. As  $\epsilon \rightarrow 0$  the velocity profile  $f'$  is close to  $-1$  except for a thin shear layer around  $\eta = \frac{1}{2}\eta_p$  where  $f'$  attains its maximum. Let the thickness of the shear layer be  $\epsilon^q$ ,  $q > 0$ . The presence of the factor  $\lambda$  in the Falkner-Skan equation suggests that the asymptotic expansion of  $f$  inside the shear layer proceeds in integer powers of  $\epsilon$ . Further, as the viscous term  $f'''$  is likely to play an important role, the asymptotic expansion in the shear layer is chosen as

$$f(\eta) = \epsilon^{-q} \{F_1(\sigma) + \epsilon F_2(\sigma) + O(\epsilon^2)\}, \tag{15}$$

where  $\sigma = \epsilon^{-q}(\eta - \frac{1}{2}\eta_p)$ . The functions  $F_1$  and  $F_2$  satisfy

$$F_1''' + F_1 F_1'' + F_1'^2 = 0, \tag{16a}$$

$$F_2''' + F_1 F_2'' + 2F_1' F_2' + F_1'' F_2 = -F_1'^2. \tag{16b}$$

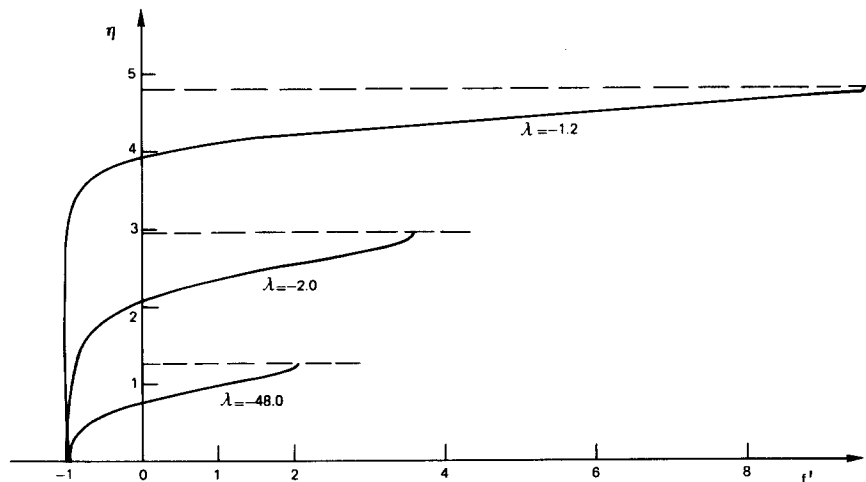


Figure 7. Periodic solutions of the Falkner-Skan equation; half a period is shown.

Table 5. Properties of the periodic Falkner-Skan solution, compared with asymptotic theory for  $\lambda \rightarrow -1^-$ .

$\lambda$	period $\eta_p$		$f'_{\max}$	
	numerical	Eq. (20)	numerical	Eq. (19)
-1.00030	200.06	199.99	5001.4	4999.50
-1.00300	63.47	63.21	501.6	499.50
-1.03030	20.69	19.90	51.62	49.50
-1.09981	12.32	10.96	17.15	15.03
-1.2	9.53	7.75	9.62	7.50
-2.0	5.90	3.46	3.59	1.50
-6.0	3.389		2.331	
-18.0	1.983		2.098	
-48.0	1.221		2.035	
-198.0	0.603		2.007	

The odd solution of (16a) we are interested in is given by

$$F_1(\sigma) = C \tanh\left(\frac{1}{2}C\sigma\right), \quad (17)$$

where  $C$  is still arbitrary. Since  $F_1'(-\infty) = 0$ , it cannot be matched to the outer region where  $f' \approx -1$ . But as we will show next,  $F_2'$  can be matched. Hereto we have to choose  $q = \frac{1}{2}$ , as is apparent from (15). Also  $C$  can be determined. Restricting ourselves to odd solutions (16b) can be integrated to

$$F_2'' + F_1 F_2' + F_1' F_2 = - \int_0^\infty F_1'^2 d\sigma. \quad (18)$$

Requiring  $F_2'(-\infty) = -1$  and  $F_2''(-\infty) = 0$ , we can derive from (17) and (18)

$$C = \int_0^\infty F_1'^2 d\sigma = \sqrt{3}.$$

Combining (15) and (17) it follows that the maximum value of  $f'$  asymptotically satisfies

$$f'_{\max} \sim \frac{1}{2} C^2 \epsilon^{-1} = \frac{3}{2} (-1 - \lambda)^{-1}. \quad (19)$$

Finally, by matching  $F_1(-\infty)$  with  $f$  in the outer region, which approximately equals  $-\eta$ , the period  $\eta_p$  is found to behave as

$$\eta_p \sim 2C\epsilon^{-1/2} = 2\sqrt{3}(-1 - \lambda)^{-1/2}. \quad (20)$$

A comparison between the asymptotes (19) and (20) and the numerical results is presented in Table 5 as mentioned before; the agreement found confirms the asymptotic behaviour.

Also in the other limit,  $\lambda \rightarrow -\infty$ , the periodic solution can be pursued. Its limiting form is given by

$$\hat{f}'(s) = 2 - 3 \tanh^2 |2^{-1/2}s|,$$

where the notation from (10) has been used.

## 7. Concluding remarks

In this paper solutions of the Falkner-Skan equation satisfying either  $f'(\infty) = 1$  (exp.) or  $f'(\infty) = -1$  are discussed. It is shown that for  $\lambda \rightarrow 0^-$  these two families of solutions are closely related, because the first terms in the two asymptotic expansions of  $f''(0)$  as  $\lambda \rightarrow 0^-$  are the same. For  $\lambda < -1$  the two types of problems both allow multiple solutions:

- (i) The solutions with  $f'(\infty) = -1$  are distinguished by  $N$ , the number of zeros of  $f' - 1$ , which takes the values 0, 2, 4, . . . . The branches with  $N > 2$  are found to lie very close together, suggesting the existence of a limit curve of solution branches with  $f'(\infty) = -1$  as  $N$  increases indefinitely (Figure 6).
- (ii) The multiple solutions with  $f'(\infty) = 1$  may be distinguished also by the number  $N$ , which now takes the values 1, 2, 3, . . . . The first seven of these Libby and Liu branches have been continued to larger values of  $\lambda$ . We have found all these branches to converge to the limit curve of solutions with  $f'(\infty) = -1$  for sufficiently large  $\lambda < -1$ .

From these numerical results it is observed that the branching structure is dominated by the point  $B = (-1, -1.08638)$ . Due to the ill-posedness of this problem as mentioned earlier no precise structure of this branching process has been found. There is a definite possibility that  $B$  is one giant branching point from which all branches start. Such a structure would suggest the existence of a limit curve representing the branch of solutions which oscillate infinitely many times. This limit curve, which extends from  $\lambda = -1$  to  $\lambda = -\infty$ , would then separate the solutions with  $f'(\infty) = 1$  from those with  $f'(\infty) = -1$ . In favour of the existence of such a limit curve is the observation of periodic solutions for  $\lambda < -1$ . Further analysis would be appropriate if one intends to reveal the precise structure of this branching process.

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